Substitutions, Tilings and Partitions

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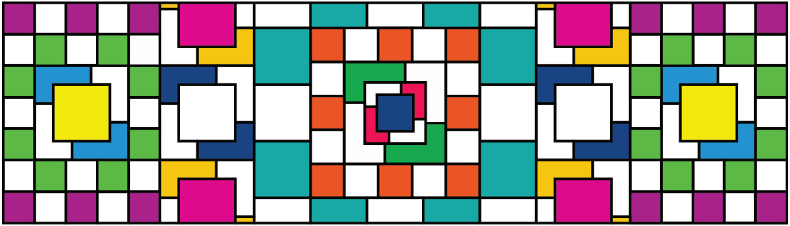
Department of Mathematics Undergraduate Colloquium, University of Houston



Tiles and tilings

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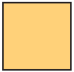
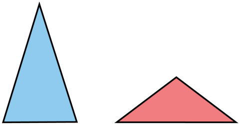
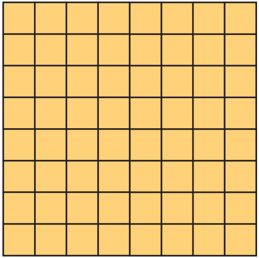
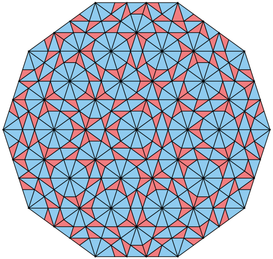




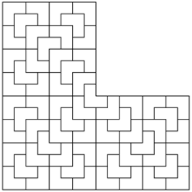
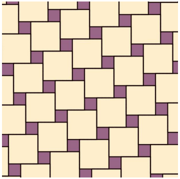
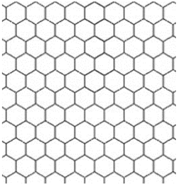
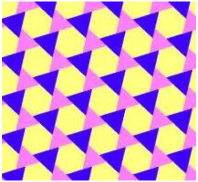
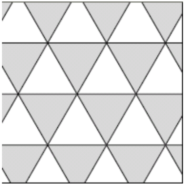
A tiling is a union of tiles which covers $\mathbb{R}^d$, and different tiles can intersect only at the boundaries:





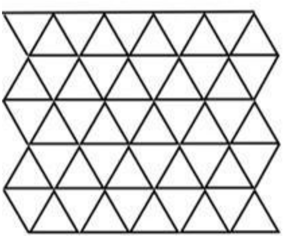
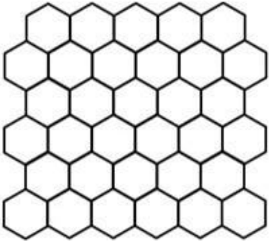
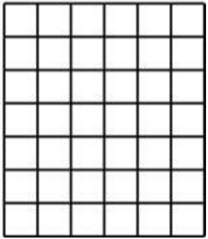
More tilings

Tilings with finitely many tiles up to translations

Periods

A period is a vector $v \in \mathbb{R}^d$ such that $\tau + v = \tau$

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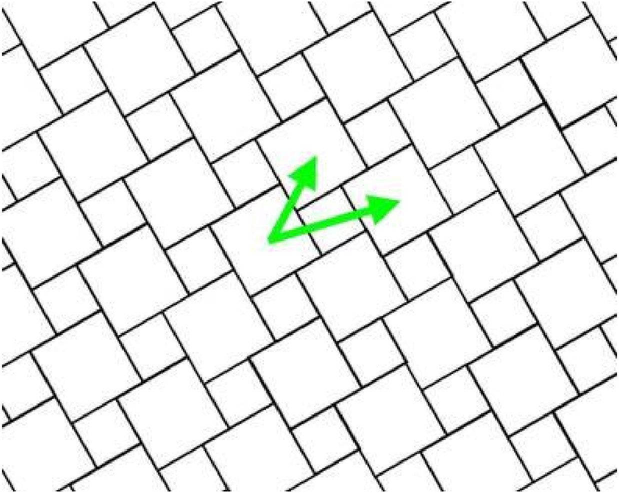
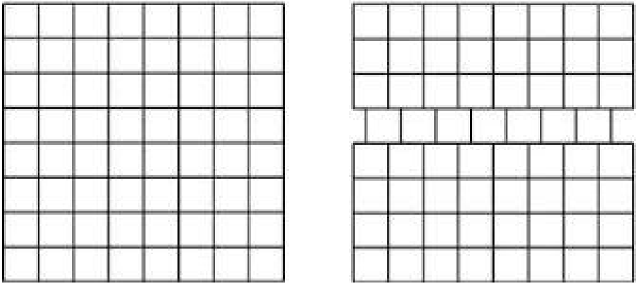
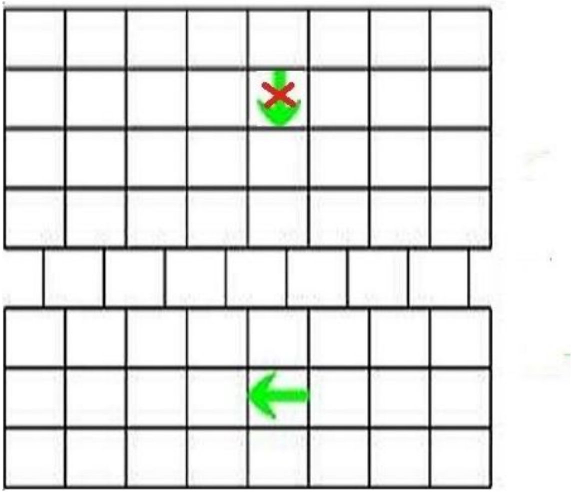


Shifting a row in the grid



We now have periods in one direction only



Non-periodicity

A tiling in $\mathbb{R}^d$ is called periodic or strongly periodic if there exist d linearly independent periods.

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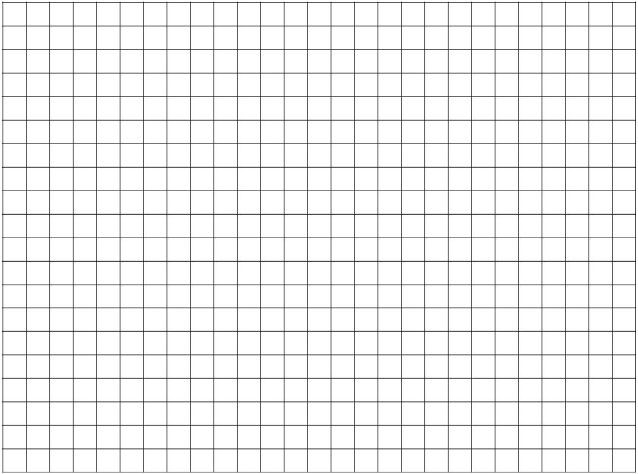
Silly example - local symmetry break



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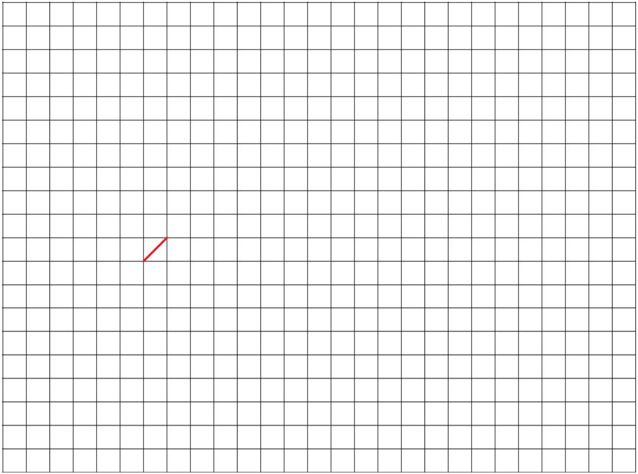

Interesting questions

Are there tilings which are non-periodic but have no local symmetry break?

That is - can a tiling that "looks the same" everywhere, namely does not exist a place which by looking at a local neighborhood you can say exactly where you are, be non periodic?

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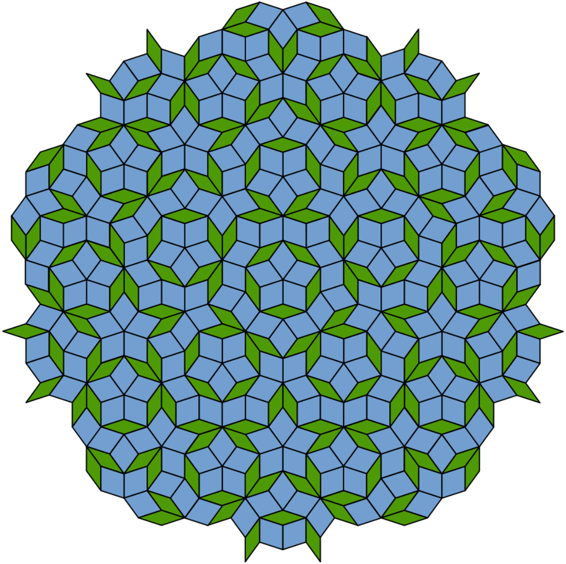
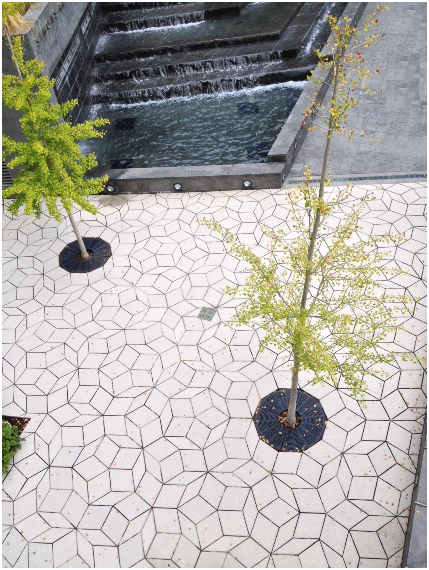
Is there a finite set of tiles that can tile $\mathbb{R}^d$ **only** in non-periodically?

Penrose says YES!!!



And he should know

More interesting floors



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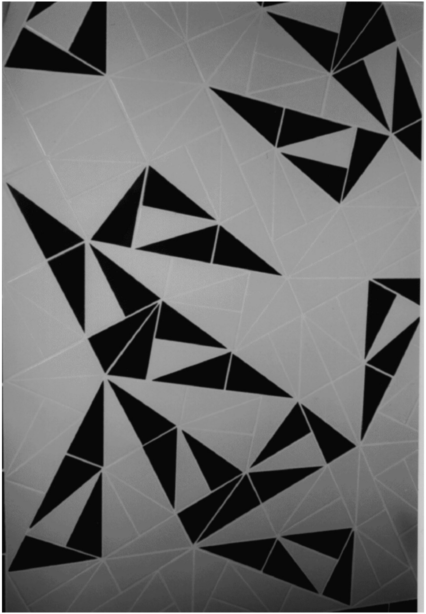

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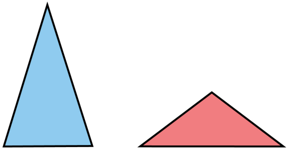

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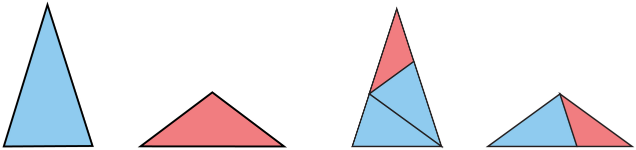

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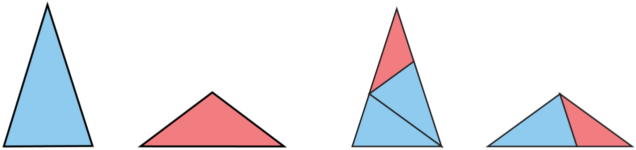
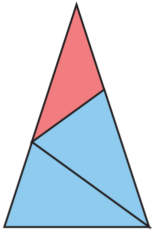


After the substitution - inflate



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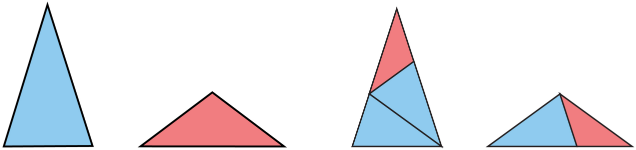
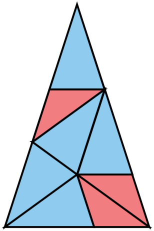


After the substitution - inflate, and substitute again...



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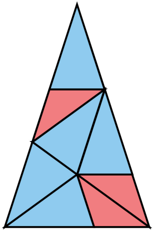

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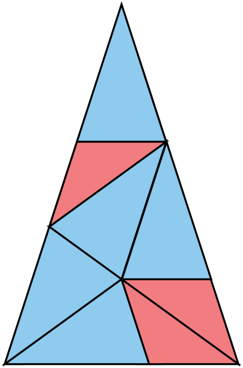

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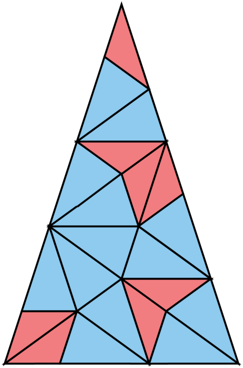

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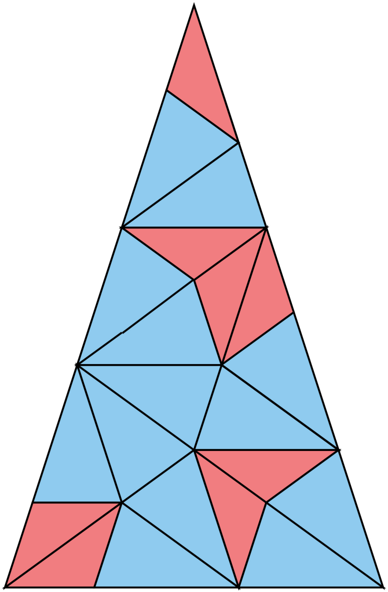

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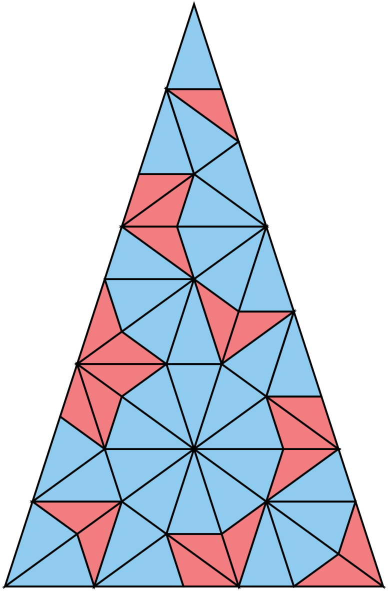

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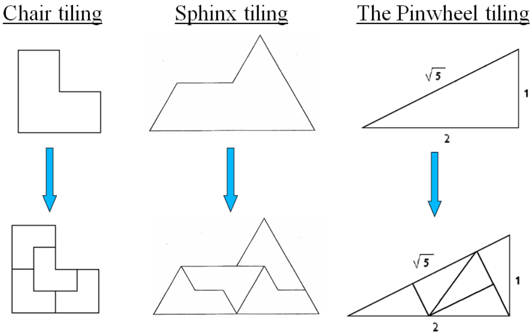
In any case in any such tiling every pattern which appears is a translation of a sub-pattern of one of the tilings of the finite domains described in the process.

More examples of substitutions



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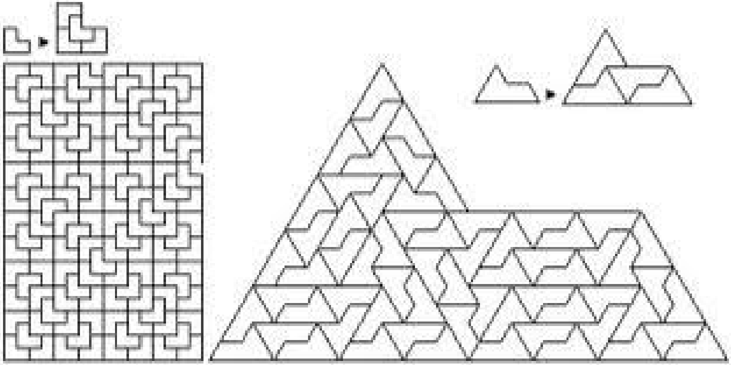

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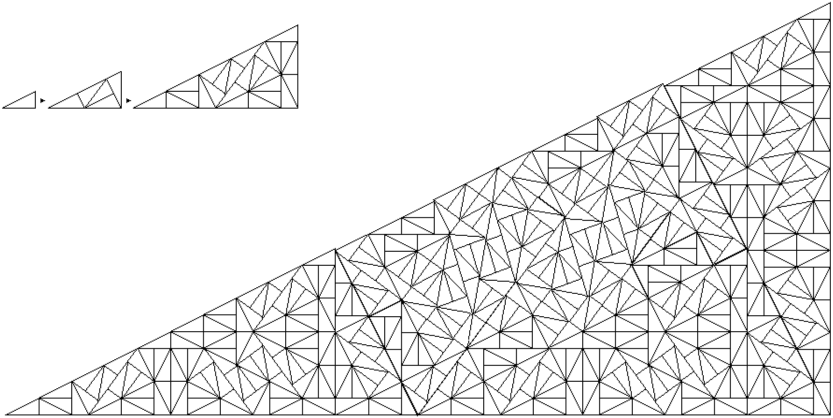

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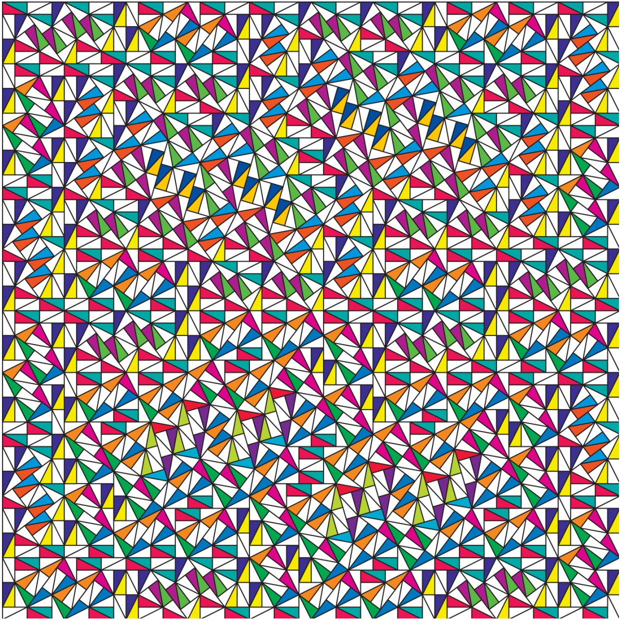

Multiscale substitution schemes

We can expand our definitions, and allow more than one scale to appear in the substitution rule

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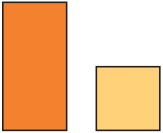

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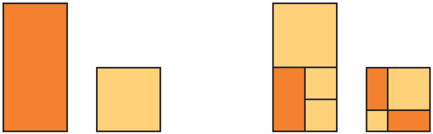

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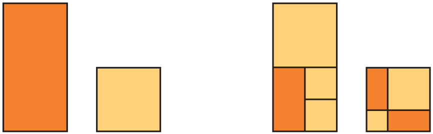

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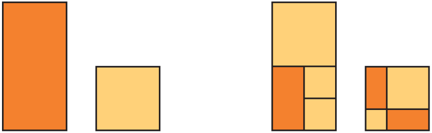

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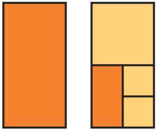

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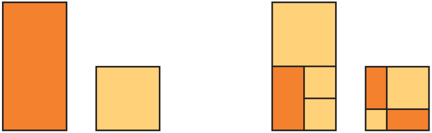

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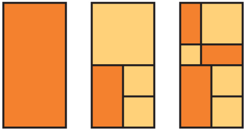

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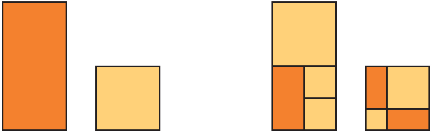

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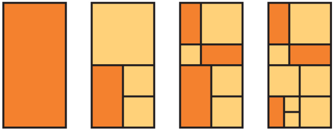

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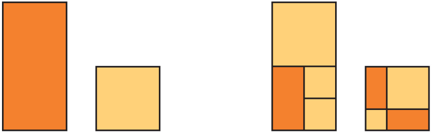

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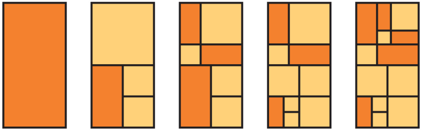

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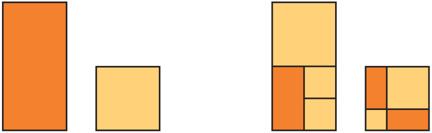

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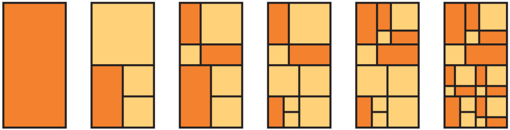

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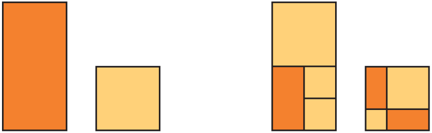

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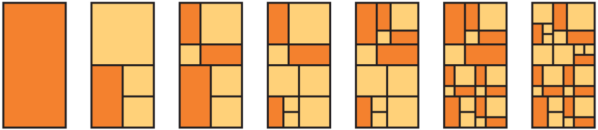

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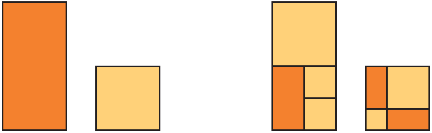

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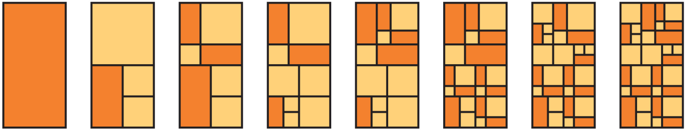

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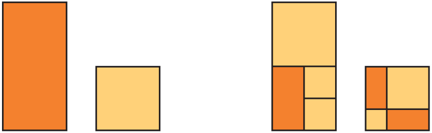

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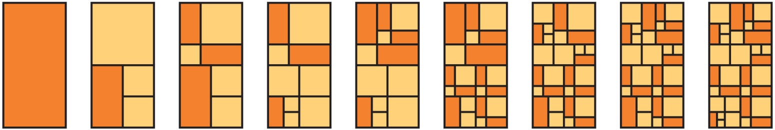

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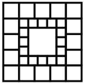

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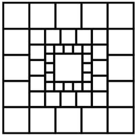

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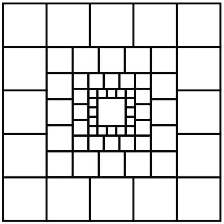

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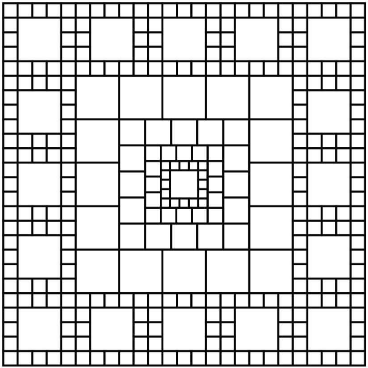

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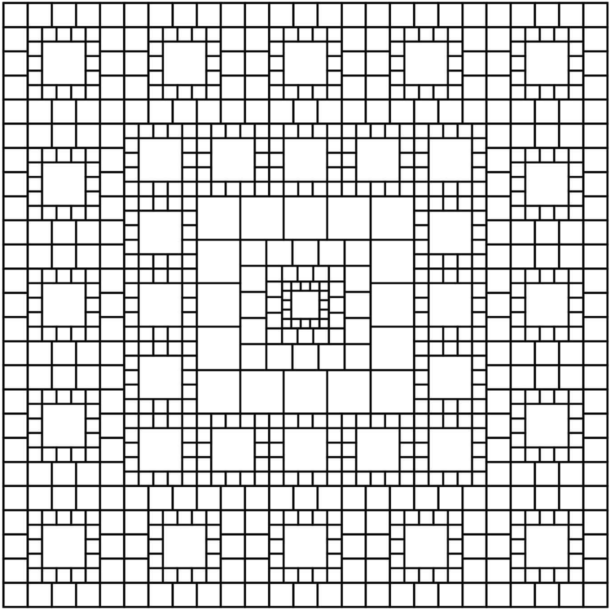

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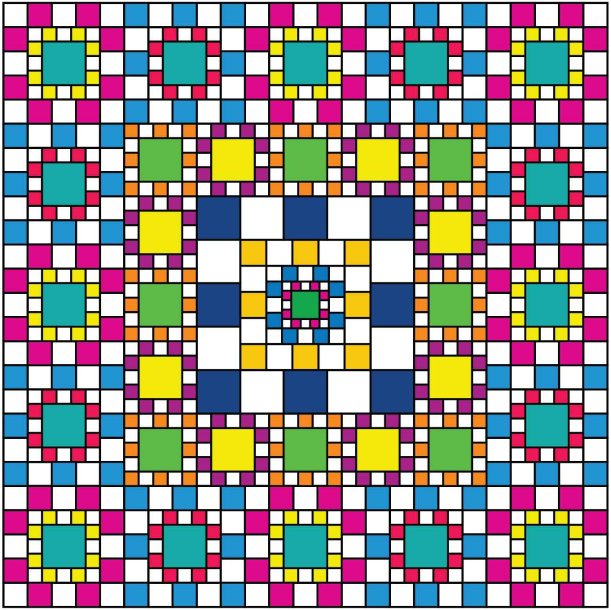

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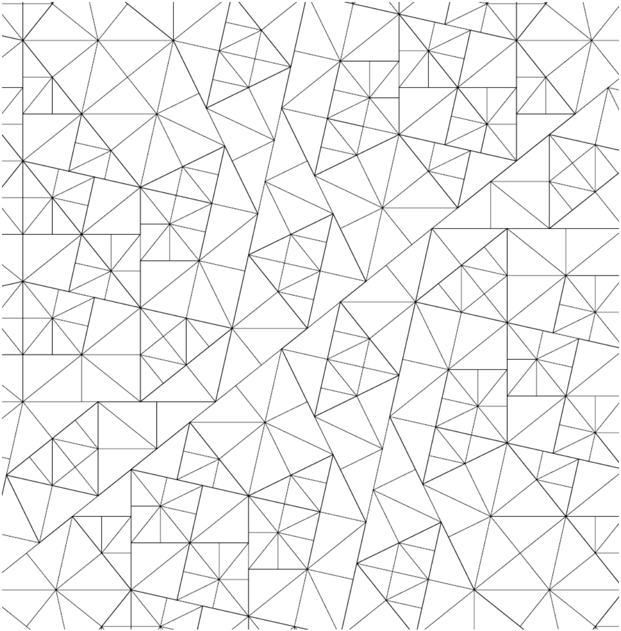
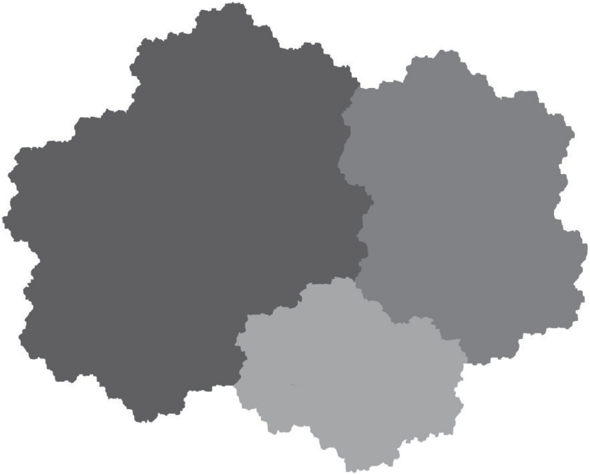


Sometimes they can define fractals



Scales are τ, τ^2, τ^3, where $\tau + \tau^2 + \tau^3 = 1$.

Kakutani sequences

The scheme illustrated by

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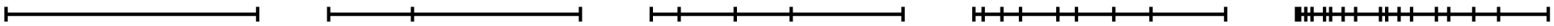

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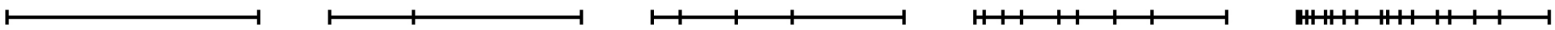

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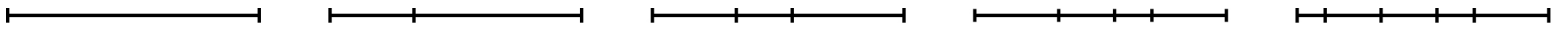

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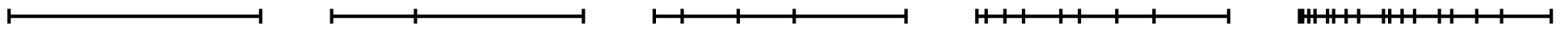

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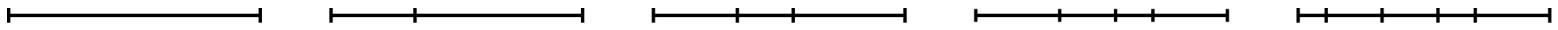

The first sequence is not nicely distributed, but the second one is (this is not a trivial fact..)

A nice question

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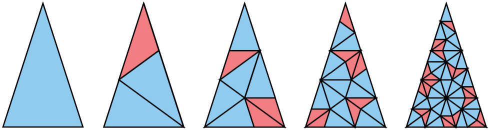
3. In case the limits exist, are they necessarily the same? No!

Substitution matrix



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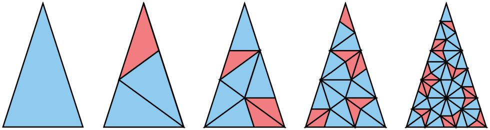

Let a_n, b_n be the number of blue and red triangles in the nth iteration.

$$\begin{pmatrix} a_0 \\ b_0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} a_1 \\ b_1 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}, \quad \begin{pmatrix} a_2 \\ b_2 \end{pmatrix} = \begin{pmatrix} 5 \\ 3 \end{pmatrix}$$

$$\begin{pmatrix} a_3 \\ b_3 \end{pmatrix} = \begin{pmatrix} 8 \\ 5 \end{pmatrix}, \quad \begin{pmatrix} a_4 \\ b_4 \end{pmatrix} = \begin{pmatrix} 13 \\ 8 \end{pmatrix}, \quad \ldots$$

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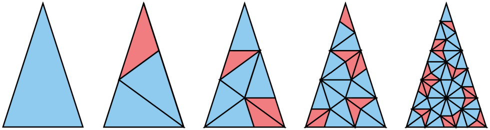

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So $\begin{pmatrix} a_n \\ b_n \end{pmatrix} = \begin{pmatrix} F_{n+1} \\ F_n \end{pmatrix}$, where F_n is the Fibonacci sequence $F_n = 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, \ldots$!

Ratio between types of tiles

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Showing that in fact there are no periods at all is slightly more difficult, but it is definitely true!

Just ask Penrose!

Thanks!